CELLULAR SUBSETS OF THE 3-SPHERE

BY

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1. **Introduction.** A subset K of S^3 is called *cellular* if there is a sequence $\{C_i\}_{i=1}^{\infty}$ of 3-cells such that $C_{i+1} \subset \operatorname{Int} C_i$ and $K = \bigcap_{i=1}^{\infty} C_i$. The concept of cellularity was defined by M. Brown [5] in 1960. The purpose of this paper is to study the relationship between cellularity of a set and the cellularity of certain subsets of the set. The sets of primary interest are cells, but some results are obtained concerning a wider class of continua.

A subset K of S^3 is called *pointlike* if $S^3 \setminus K$ is homeomorphic to E^3. The equivalence of being cellular and of being pointlike is shown in Lemma 2.2. Hence cellularity of a set will be established by showing that its complement is an open 3-cell.

The main results are found in §§3, 4, and 5 which deal with arcs, disks, and 3-cells, respectively. An example is given in §6 of a cellular arc which contains no tame subarc. §2 is devoted to preliminary results and operational lemmas.

An *n-manifold* M is a connected Hausdorff space with a countable basis, the closure of each basis element being an n-cell. The set of points of M which have open n-cell neighborhoods is designated by $\operatorname{Int} M$, while $\operatorname{Bd} M$ denotes $M \setminus \operatorname{Int} M$. In addition to the concept of $\operatorname{Bd} M$ where M is a manifold we have the concept of the (set theoretic) *boundary* of $K \subset S^3$ which is $\operatorname{Cl} K \cap \operatorname{Cl}(S^3 \setminus K)$. The symbol $V(X, \epsilon)$ denotes the set of all points in S^3 whose distance from X is less than ϵ. By a continuum we mean a compact connected set.

2. **Preliminaries.** The following lemma, which is a direct consequence of Theorem 2 of [4], characterizes those subsets of S^3 which are open 3-cells.

LEMMA 2.1. *Let U be an open connected subset of S^3 which has a connected boundary. Suppose that each polygonal simple closed curve in U lies in a 3-cell in U. Then, U is an open 3-cell.*

REMARK. Since S^3 is unicoherent the complement of a continuum which does not separate S^3 will have a connected boundary.

The following shows the equivalence of the concepts of being cellular and of being pointlike.

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LEMMA 2.2. *A subset of S^3 is cellular if and only if it is pointlike.*

Proof. Let K be a cellular subset of S^3, and let J be a polygonal simple closed curve in $S^3\backslash K$. It will suffice to show that there is a polyhedral 2-sphere T which separates K from J. This would show that J lies in a 3-cell in $S^3\backslash K$ and hence show that $S^3\backslash K$ is an open 3-cell.

Since K is cellular there is a 3-cell C such that $K \subset \operatorname{Int} C$ and $J \cap C = \square$. Let ϵ be a positive number which is less than each of $\rho(\operatorname{Bd} C, J)$ and $\rho(\operatorname{Bd} C, K)$. By [3] there is a homeomorphism h of $\operatorname{Bd} C$ such that h moves no point more than ϵ and $h(\operatorname{Bd} C)$ is polyhedral. Then, $h(\operatorname{Bd} C)$ is the desired 2-sphere, and hence K is pointlike.

Now suppose that K is pointlike and let h be a homeomorphism of E^3 onto $S^3\backslash K$. For each positive integer n let T_n be the 2-sphere defined by $x^2+y^2+z^2=n^2$. If $S_n = h(T_n)$, then S_n has a shell neighborhood in S^3. Let C^n be the closure of the complementary domain of S_n which contains K. It follows from [5] that C_n is a 3-cell, $C_{n+1} \subset \operatorname{Int} C_n$, and $K = \bigcap_{i=1}^{\infty} C_n$. Therefore K is cellular.

As a direct result of this lemma and [6] we have the following:

LEMMA 2.3. *Suppose that for each i, K_i is cellular and $K_{i+1} \subset K_i$. Then, $K = \bigcap_{i=1}^{\infty} K_i$ is cellular.*

In general the union of two cellular sets is not cellular, even if the complement of the union is connected and simply connected (see [8, Example 1.3]). In order to determine when the union of two cellular sets is cellular, it is first necessary to determine some conditions which when imposed on certain subsets of a set insure the cellularity of that set. To facilitate this the following operational lemmas are needed.

LEMMA 2.4. *Let A and B be disjoint continua in S^3, and let T be a 2-sphere which separates these sets. Let D be a disk in $S^3\backslash(A \cup B)$ such that $D \cap T = \operatorname{Bd} D = J$. Then, if the disks on T determined by J are E_1 and E_2, either $E_1 \cup D$ or $E_2 \cup D$ separates A from B.*

Proof. Let U_1 and U_2 be the complementary domains of T, and assume that $A \subset U_1$ and $\operatorname{Int} D \subset U_1$. Let V_1 be the component of $U_1\backslash D$ containing A. The boundary of V_1 will be either $E_1 \cup D$ or $E_2 \cup D$, and it will separate A from B.

LEMMA 2.5. *Let A and B be disjoint continua in S^3, and let $D_1, D_2, \cdots, D_n$ be a collection of pairwise disjoint polyhedral disks whose interiors lie in $S^3\backslash(A \cup B)$. Suppose that there is a polyhedral 2-sphere T separating A from B such that $T \cap \operatorname{Bd} D_i = \square$ $(i = 1, 2, \cdots, n)$. Then, there is a polyhedral 2-sphere T' separating A from B such that $T' \cap (\bigcup_{i=1}^{n} D_i) = \square$.*

Proof. Since each of $D_1, D_2, \cdots, D_n$, and T is polyhedral, there is no loss of generality in assuming that $T \cap (\bigcup_{i=1}^n D_i)$ consists of a finite number of simple closed curves. Let $\mathscr{T}$ be the collection of all polyhedral 2-spheres T such that

(1) T separates A from B,

(2) $T \cap \mathrm{Bd}\, D_i = \square$ $(i = 1, 2, \cdots, n)$, and

(3) $T \cap (\bigcup_{i=1}^n D_i)$ consists of a finite number of simple closed curves.

Pick $T' \in \mathscr{T}$ such that if $T_1 \in \mathscr{T}$ then the number of components of $T_1 \cap (\bigcup_{i=1}^n D_i)$ is not less than the number of components of $T' \cap (\bigcup_{i=1}^n D_i)$. The result will be obtained by showing that the latter number is zero.

Suppose that $T' \cap (\bigcup_{i=1}^n D_i) \neq \square$, and let $\mathscr{J}$ be the collection of components of this intersection. Let $\mathscr{D}$ be the collection of subdisks of the disks $\{D_i\}_{i=1}^n$ such that $E \in \mathscr{D}$ if and only if $\mathrm{Bd}\, E \in \mathscr{J}$. Let the elements of $\mathscr{D}$ be partially ordered by inclusion. Since this collection is finite, there is an element $E_1 \in \mathscr{D}$ such that E_1 contains no other element of $\mathscr{D}$. Let $J_1 = \mathrm{Bd}\, E_1$. Since J_1 divides T' into disks F_1 and F_2, it follows from Lemma 2.4 that one of the spheres $E_1 \cup F_i$ $(i = 1, 2)$, say $E_1 \cup F_1$, separates A from B. By deforming $F_1 \cup E_1$ away from E_1, we obtain a 2-sphere $T_1 \in \mathscr{T}$ such that $T_1 \cap (\bigcup_{i=1}^n D_i)$ has fewer components than does $T' \cap (\bigcup_{i=1}^n D_i)$. This gives a contradiction. Hence $T' \cap (\bigcup_{i=1}^n D_i) = \square$.

A subset $K \subset S^3$ is said to be *locally polyhedral at* $p \in K$ if there is a neighborhood U of p such that $(\mathrm{Cl}\, U) \cap K$ is a polyhedron. If L is a subset of K and K is locally polyhedral at each point of $K \backslash L$, then K is said to be *locally polyhedral* $\mathrm{mod}\, L$.

Let A, B, and D be continua in S^3 such that $A \subset B \subset D$ and A can be imbedded in the plane. The statement that B has the *expansion property relative to* D $\mathrm{mod}\, A$ means that for $\epsilon > 0$, there is a continuum C such that

(1) $B \subset C \subset V(B, \epsilon)$,

(2) C is bounded by a 2-sphere T,

(3) $T \cap D = A$, and

(4) T is locally polyhedral $\mathrm{mod}\, A$.

The following example will illustrate the above definition. Let D be the union of two cells, B and B', each of dimension $\leqq 3$ but not necessarily the same dimension, such that $B \cap B' = (\mathrm{Bd}\, B) \cap (\mathrm{Bd}\, B') = A$ where A is a cell. Let B be locally polyhedral $\mathrm{mod}\, A$. Then B has the expansion property relative to D $\mathrm{mod}\, A$.

THEOREM 2.1. *Let A be a continuum in S^3, and let A' be a cellular subset of A. Suppose that there exist continua $K_1, \cdots, K_n$, in A such that* $\mathrm{Cl}(A \backslash A') = \bigcup_{i=1}^n K_i$ *and such that for each i,*

(1) *$K_i \cap A' = L_i$ is a continuum which fails to separate the plane,*

(2) K_i *has the expansion property relative to* $A \bmod L_i$, *and,*
(3) *for* $i \neq j$, $K_i \cap K_j = \square$.
Then, A *is cellular.*

Proof. Let us first show that $S^3 \backslash A$ is connected. Then in view of the remark following Lemma 2.1, $S^3 \backslash A$ will have a connected boundary. Let p and q be distinct points in $S^3 \backslash A$, and let α be a polygonal arc in $S^3 \backslash A'$ whose endpoints are p and q. This is possible since A' is cellular. Since each K_i has the expansion property, we expand them to get continua $C_1, \cdots, C_n$, each bounded by a 2-sphere which is locally polyhedral $\bmod L_i$. We may assume that $\alpha \cap (\bigcup_{i=1}^n C_i)$ consists of a finite number of pairwise disjoint arcs. Since each L_i fails to separate the plane, the arcs of $\alpha \cap (\bigcup_{i=1}^n C_i)$ may be replaced by arcs which lie on the boundary spheres of the C_i's to give an arc β in $S^3 \backslash A$ which joins p and q. Hence $S^3 \backslash A$ is connected.

Now let J be a polygonal simple closed curve in $S^3 \backslash A$, and let T be a polyhedral 2-sphere which separates A' from J. Since the K_i's have the expansion property and are pairwise disjoint, we can expand them to get continua $C_1, \cdots, C_n$, which are bounded by the 2-spheres $T_1, \cdots, T_n$, respectively, and such that T_i is locally polyhedral $\bmod L_i$. This expansion can be taken small enough so that each C_i misses J and, since the K_i's are pairwise disjoint, so that the C_i's are pairwise disjoint.

Since L_i fails to separate the plane there are polyhedral disks $D_1, \cdots, D_n$, such that $D_i \subset (T_i \backslash L_i)$ and $T_i \cup T \subset \operatorname{Int} D_i$. Let E_i be the disk on T_i complementary to D_i. By Lemma 2.5 there is a polyhedral 2-sphere T' such that T' separates $A \cup (\bigcup_{i=1}^n E_i)$ from J and $T' \cap (\bigcup_{i=1}^n D_i) = \square$. Therefore T' separates A from J, and this establishes the cellularity of A.

COROLLARY 2.1. *Let* A *be a continuum in* S^3, *and let* A' *be a cellular subset of* A. *Suppose that* A *has the expansion property relative to* $A \bmod A'$. *Then* A *is cellular.*

COROLLARY 2.2. *Let* A *and* B *be continua in* S^3 *such that*
(1) A *is cellular,*
(2) $A \cap B$ *is a continuum which fails to separate the plane, and*
(3) B *has the expansion property relative to* $(A \cup B) \bmod (A \cap B)$.
Then, $A \cup B$ *is cellular.*

COROLLARY 2.3. *Let* α *be an arc and* $K \subset \alpha$. *Suppose that* α *is locally polyhedral* $\bmod K$. *Then, if* K *is either a point or a cellular arc,* α *is cellular.*

COROLLARY 2.4. *Let* T *be a 2-sphere, and let* K *be a cellular subset of* T. *Let* U_1 *and* U_2 *be the complementary domains of* T. *Suppose that* T *is locally polyhedral* $\bmod K$. *Then,* U_i *is an open 3-cell* $(i = 1, 2)$.

Some results could be stated here which concern disks, but since they are included in more general results in §4, they will not be given here.

3. **Cellular sets of dimension one.** The sets of primary interest in this section are arcs, but some results are obtained which concern more general one-dimensional sets.

LEMMA 3.1. *Let T be a polyhedral 2-manifold without boundary, and let K be a compact subset of T. Suppose that K can be embedded in the plane and that each of its components fails to separate the plane. Then, there is a polyhedral disk D on T such that $K \subset \operatorname{Int} D$.*

Proof. Let T' be the decomposition space of T which is formed by identifying each component of K to a point. Since no component of K separates the plane, it follows from [14] that T' is a manifold. The image of K under this mapping will lie on an arc $\alpha \subset T'$. Since T' is a manifold, there is a disk $D' \subset T'$ such that $\alpha \subset \operatorname{Int} D'$. Let E be the inverse image of D' on T. Since T is polyhedral and $K \subset \operatorname{Int} E$, a slight adjustment of $\operatorname{Bd} E$ will give the desired polyhedral disk D.

THEOREM 3.1. *Let K be a cellular set which can be embedded in the plane and such that each of its subcontinua fails to separate the plane. Then, if L is a subcontinuum of K, L is cellular.*

Proof. Let J be a polygonal simple closed curve in $S^3 \backslash L$. Since J is polygonal there is a polyhedral torus T which separates J from L. Let $A = K \cap T$. Since no component of A separates the plane, there is a polyhedral disk $D \subset T$ such that $A \subset \operatorname{Int} D$.

Let $B = T \backslash \operatorname{Int} D$. Since K is cellular, B is a continuum, and $K \cap B = \square$, there is a polyhedral 2-sphere S which separates K from B. Then S must therefore separate L from B. Since $S \cap \operatorname{Bd} D = \square$, there is a polyhedral 2-sphere S' such that S' separates L from B and $S' \cap D = \square$. It follows that S' separates L from J, and hence L is cellular.

COROLLARY 3.1. *Let K be a dendrite or a pseudo-arc, and suppose that K is cellular. Then, each subcontinuum of K is cellular.*

Let α be an arc, and let p be a point of α. The statement that α has *penetration index k at the point p*, in symbols $P(\alpha,p) = k$, means that k is the smallest positive integer such that there are arbitrarily small 2-spheres enclosing p and containing no more than k points of α. This definition is due to Alford and Ball [2].

THEOREM 3.2. *Let α be an arc, and let p be an endpoint of α. Suppose that $P(\alpha,p) = 1$ and that each arc of $\alpha \backslash \{p\}$ is cellular. Then, α is cellular.*

Proof. Let J be a polygonal simple closed curve in $S^3\backslash\alpha$, and let T be a 2-sphere separating J from p such that $T\cap\alpha=\{q\}$. We may assume that T is locally polyhedral mod $\{q\}$. Let r be the endpoint of α distinct from p. The subarc $rq=\beta\subset\alpha$ is cellular, and hence there is a polyhedral 2-sphere T_1 which separates β from J.

Let D be a polyhedral disk on $T\backslash\{q\}$ such that $T_1\cap T\subset \operatorname{Int} D$. Let $E=T\backslash\operatorname{Int} D$. Since $E\cup\beta$ is a continuum, it follows from Lemma 2.5 that there is a polyhedral 2-sphere T' which separates J from $E\cup\beta$ and such that $D\cap T'=\square$. It follows that T' separates α from J, and therefore α is cellular.

THEOREM 3.3. *Let α be an arc with endpoints p and q. Suppose that $P(\alpha,p)=P(\alpha,q)=1$ and that each arc of* $\operatorname{Int}\alpha$ *is cellular. Then, α is cellular.*

Proof. Let β be a subarc of α which does not contain p. If $\beta\subset\operatorname{Int}\alpha$, then β is cellular by hypothesis. If $q\in\beta$, then by Theorem 3.2 β is cellular, since $P(\beta,q)=P(\alpha,q)=1$. The cellularity of α now follows from Theorem 3.2.

4. Cellular disks.

LEMMA 4.1. *Let D be a disk, and let K be a subcontinuum of D which does not separate the plane. Suppose that $K\cap\operatorname{Bd} D\neq\square$ and D is locally polyhedral* mod K. *Then, if $\epsilon>0$, there is a subdisk D' of D such that*

(1) $K\subset D'\subset V(K,\epsilon)$,

(2) $\operatorname{Cl}(D\backslash D')\cap K=\square$, *and*

(3) $\operatorname{Cl}(D\backslash D')$ *consists of a finite number of pairwise disjoint polyhedral disks, each meeting D' in an arc.*

Proof. For each point $p\in K\backslash\operatorname{Bd} D$ let D_p be a subdisk of $\operatorname{Int} D$ such that

(1) $p\in\operatorname{Int} D_p$,

(2) $D_p\subset V(p,\epsilon)$, and

(3) D_p is locally polyhedral mod $(K\cap D_p)$.

For each $p\in K\cap\operatorname{Bd} D$ let D_p be a subdisk of D such that

(1) $D_p\subset V(p,\epsilon)$,

(2) D_p is the closure of an open (relative to D) set containing p,

(3) $D_p\cap\operatorname{Bd} D$ is an arc, and

(4) D_p is locally polyhedral mod $(K\cap D_p)$.

Let $D_1,D_2,\cdots,D_n$ be a finite subcollection of these disks such that K is covered by the open (relative to D) sets whose closures are the D_i's. Let $M=\bigcup_{i=1}^n D_i$. With no loss of generality M may be assumed to be a manifold with a finite number of boundary components. If this were not the case, a manifold could be obtained by slight adjustments of the boundaries of the D_i's. So M must be a disk with holes in it. Let J be the boundary component of M which bounds a disk $E \subset D$ such that $M \subset E$. Since K fails to separate the plane, each boundary component of M distinct from J may be connected to J by an arc which lies in $M \backslash K$. By removing neighborhoods of these arcs from M we obtain a disk D' satisfying conditions (1), (2), and (3).

THEOREM 4.1. *Let D be a disk, and let K be a cellular subset of D. Suppose that D is locally polyhedral* mod K. *Then, D is cellular.*

Proof. Suppose first that $K \cap \mathrm{Bd}\, D \neq \square$. Let J be a polygonal simple closed curve in $S^3 \backslash D$, and let T be a polyhedral 2-sphere which separates K from J. By Lemma 4.1 there is a subdisk D' of D containing K and separated from J by T such that $\mathrm{Cl}(D \backslash D')$ consists of a finite collection of pairwise disjoint polyhedral disks. Furthermore each disk will meet D' in an arc. Let these disks be $D_1, D_2, \cdots, D_n$, and let $D_i \cap D' = \alpha_i$.

Since each D_i is polyhedral there exist pairwise disjoint polyhedral 3-cells, $C_1, C_2, \cdots, C_n$, in $S^3 \backslash J$ such that for each i, $D_i \subset C_i$ and $(\mathrm{Bd}\, C_i) \cap D = \alpha_i$. We may assume that for each i, $T \cap \mathrm{Bd}\, C_i$ consists of a finite number of simple closed curves. For each i let E_i be a disk on $\mathrm{Bd}\, C_i$ such that $T \cap \mathrm{Bd}\, C_i \subset \mathrm{Int}\, E_i$ and $E_i \cap \alpha = \square$. Let F_i be the disk on $\mathrm{Bd}\, C_i$ which is complementary to E_i. Then T separates J from $D' \cup (\bigcup_{i=1}^n F_i)$. By Lemma 2.5 there is a polyhedral 2-sphere T' which separates J from $D' \cup (\bigcup_{i=1}^n F_i)$ and misses each E_i. Then T' separates J from D, and hence D is cellular.

Now suppose that $K \subset \mathrm{Int}\, D$. Let α be an arc on D joining $\mathrm{Bd}\, D$ and K such that $\alpha \cap K$ consists of a single point. This can be done by running an arc on D from $x \in \mathrm{Bd}\, D$ to $y \in K$ where x and y are arbitrary. By parametrizing α so that x is the image of 0 and y is the image of 1, there will be a first point of $\alpha \cap K$. Call this point z. There is no loss of generality in assuming that the arc xz is locally polyhedral mod $\{z\}$. The union of K and the arc xz is cellular by Corollary 2.2. The above argument will establish the cellularity of D.

The following lemma and theorem might be considered as converses to Theorems 2.1 and 4.1, respectively.

LEMMA 4.2. *Let K be a cellular set in S^3, and let L be a subcontinuum of K. Suppose that for each open set U containing L there is a finite collection of pairwise disjoint polyhedral 2-spheres $T_1, T_2, \cdots, T_n$ such that*

(1) *if A is a component of $K \backslash U$, there is a T_i which separates A from L,*

(2) *for each i, $T_i \cap K$ is a proper subset of T_i, and*

(3) *no component of $T_i \cap K$ separates T_i.*

Then, L is cellular.

Proof. Let J be a polygonal simple closed curve in $S^3 \backslash L$ and let U be an open set such that $L \subset U$ and $U \cap J = \square$. Let $T_1, T_2, \cdots, T_n$ be a finite collection of pairwise disjoint polyhedral 2-spheres which satisfy conditions (1), (2), and (3). There is no loss of generality in assuming that for $i \neq j$, T_i is not separated from L by T_j. It may further be assumed that for each i, $T_i \cap J$ consists of a finite number of points. If this intersection is empty for each i, then either $J \cap K = \square$ or some T_i separates J from L. In either case there would be a polyhedral 2-sphere separating L from J. Hence the only case of interest is the case in which $T_i \cap J \neq \square$ for some values of i.

Since no component of $K \cap T_i$ separates T_i, there is a polygonal simple closed curve J_i on T_i which separates $T_i \cap K$ from $T_i \cap J$. Let $T_i = E_i \cup D_i$ where E_i and D_i are disks such that

(1) $\operatorname{Bd} D_i = \operatorname{Bd} E_i = J_i$, and

(2) $T_i \cap J \subset \operatorname{Int} E_i$.

Let $\alpha_1, \alpha_2, \cdots, \alpha_m$ be the subarcs of J whose endpoints are on $\bigcup_{i=1}^n T_i$ and whose interiors are not separated from L by these spheres. Let $N = (\bigcup_{i=1}^m \alpha_i) \cup (\bigcup_{j=1}^l E_{k_j})$ where the union of the E_{k_j}'s is taken over those E_i's which intersect J. Since $J \cup (\bigcup_{j=1}^l E_{k_j})$ is connected, N is connected and hence a continuum.

Since $N \cap K = \square$ and K is cellular, there is a polyhedral 2-sphere T which separates N from K, and hence separates L from N. By Lemma 2.5 there is a polyhedral 2-sphere T' separating L from N such that $T' \cap (\bigcup_{j=1}^l D_{k_j}) = \square$. Therefore T' separates L from J. The complement of L is connected, and hence L is cellular.

THEOREM 4.2. *Let D be a cellular disk, and let K be a subcontinuum of D which does not separate the plane. Suppose that D is locally polyhedral* $\operatorname{mod} K$. *Then, K is cellular.*

Proof. Assume first that $K \cap \operatorname{Bd} D \neq \square$, and let U be an open set containing K. By Lemma 4.1 there is a subdisk D' of D such that

(1) $K \subset D' \subset U$,

(2) $\operatorname{Cl}(D \setminus D') \cap K = \square$, and

(3) $\operatorname{Cl}(D \setminus D')$ consists of a finite number of pairwise disjoint disks, each meeting D' in a polygonal arc. Let these disks be $D_1, D_2, \cdots, D_n$, and let $D_i \cap D' = \alpha_i$. Since these disks are pairwise disjoint, $\alpha_i \subset U$ for each i. Since each D_i is polyhedral, it can be thickened to give a polyhedral 3-cell C_i such that $D \cap \operatorname{Bd} C_i = \alpha_i$. Further, these C_i's may be obtained so that they are pairwise disjoint.

Since $\alpha_i \subset U$, each component of $D \setminus U$ will be separated from K by one of the 2-spheres $\{\operatorname{Bd} C_i\}_{i=1}^n$. Therefore, by Lemma 4.2, K is cellular.

Now consider the case in which $K \subset \operatorname{Int} D$. As in Theorem 4.1 there is an arc $\alpha \subset D$ such that

(1) α joins $\mathrm{Bd}\, D$ to K,
(2) $\alpha \cap K$ consists of a single point p, and
(3) α is locally polyhedral $\mathrm{mod}\, p$.
By the above argument, $K' = K \cup \alpha$ is cellular. Then, if U is an open set containing K, $K' \backslash U$ lies on a polygonal arc β such that $\beta \cap \mathrm{Cl}(K' \backslash \beta) = \{q\}$ where q is the endpoint of β which lies in U. The above technique will show that K is cellular.

COROLLARY 4.1. *Let D be a disk, and let K be a subcontinuum of D. Suppose that K is either a disk, a dendrite, or a pseudo-arc and that D is locally polyhedral* $\mathrm{mod}\, K$. *Then, a necessary and sufficient condition that D be cellular is that K be cellular.*

In the following theorem some different conditions are imposed on subsets of a disk which will insure that the disk be cellular.

THEOREM 4.3. *Let D be a disk which is locally polyhedral* $\mathrm{mod}\, \mathrm{Bd}\, D = J$. *Suppose that for each $\epsilon > 0$ there is a polyhedral solid torus T such that $J \subset \mathrm{Int}\, T \subset T \subset V(D, \epsilon)$. Then D is cellular*(2).

Proof. Let J_1 be a polygonal simple closed curve in $S^3 \backslash D$. Let T be a polyhedral solid torus as above such that $T \subset S^3 \backslash J_1$ and J is not contractible in T. Assume that $D \cap \mathrm{Bd}\, T$ consists of a finite number of simple closed curves. We shall first find a polyhedral 2-sphere which separates J from J_1.

Consider the components of $D \cap \mathrm{Bd}\, T$ and pick an "innermost curve" (one which separates no other from $\mathrm{Bd}\, D$ on D) and call it J_2. If J_2 fails to separate $\mathrm{Bd}\, T$, then it must circle T longitudinally a positive number of times. If this were not the case then J_2 would be a meridian curve. By removing a neighborhood of D_1 (the subdisk of D bounded by J_2) from T we have J in a 3-cell in T. This is a contradiction to the above conditions imposed on T. Since J_2 circles T longitudinally we can form a 3-cell C by thickening D_1 and adding it to T. The boundary of C will be the desired 2-sphere.

If J_2 separates $\mathrm{Bd}\, T$ let D_2 be the disk on $\mathrm{Bd}\, T$ whose boundary is J_2. Let C be the 3-cell which is bounded by $D_1 \cup D_2$ and misses J. If $J_1 \subset C$ then $D_1 \cup D_2$ is the desired 2-sphere. If not, then form a new torus $T \cup C$ or $\mathrm{Cl}(T \backslash C)$ depending on whether $D_1 \cap \mathrm{Int}\, T = \square$ or $D_1 \subset T$. By deforming this torus away from D_1 we get a new solid torus T' having the above properties and such that $D \cap \mathrm{Bd}\, T'$ has fewer components than $D \cap \mathrm{Bd}\, T$.

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(2) The author is indebted to the referee for suggesting this improved version of the author's theorem.

Repeating this process we will, after a finite number of steps, obtain a 2-sphere S which separates J_1 from J. If this did not occur then all components of $D \cap \text{Bd}\, T$ would separate $\text{Bd}\, T$. Then a finite number of the above operations of adding or deleting cells would give a solid torus containing D. This would contradict the condition that J is not contractible in T. Hence there is a 2-sphere S which separates J_1 from J.

Now let D_3 be a polyhedral disk in $\text{Int}\, D$ such that $S \cap D \subset \text{Int}\, D_3$. Since S separates J_1 from $\text{Cl}(D \backslash D_3)$, we can apply Lemma 2.5 to get a polyhedral 2-sphere S_1 such that S_1 separates J_1 from D. Hence D is cellular.

5. **Cellular 3-cells.** Since any compact, proper subset of S^3 lies in a polyhedral 3-cell in S^3, it is evident that subcells of cellular 3-cells need not be cellular. In this section some conditions are given under which certain subsets of a cellular 3-cell are cellular.

THEOREM 5.1. *Let C be a cellular 3-cell, and let J be a simple closed curve on* $\text{Bd}\, C$. *Let D be a disk in C such that $D \cap \text{Bd}\, C = \text{Bd}\, D = J$, and let A and B be the closures of the components of $C \backslash D$. Suppose that A is a 3-cell which is locally polyhedral* $\text{mod}\, D$. *Then, B is cellular.*

Proof. Let U be an open set containing B. It will suffice to show that there is a polyhedral 2-sphere T which separates $C \backslash U$ from B and such that $T \cap C$ is a disk. The conclusion will then follow from Lemma 4.2.

Let E be the disk on $\text{Bd}\, C$ which is bounded by J and lies on A. Let J_1 be a polygonal simple closed curve in $\text{Int}\, E$ such that the annulus on E bounded by $J \cup J_1$ lies in U. Let F be the disk on $\text{Bd}\, A$ which is bounded by J_1 and contains D. Since A is a 3-cell we can replace F by a polyhedral disk F_1 such that $\text{Bd}\, F = \text{Bd}\, F_1$, $\text{Int}\, F_1 \subset \text{Int}\, A$, and the subset of A bounded by $F \cup F_1$ lies entirely in U.

Let F_2 be the disk on $\text{Bd}\, A$ which is bounded by J_1 and misses B. Since F_2 is polyhedral we can deform its interior to get a polyhedral disk F_3 such that $F_3 \cap C = J_1$. The 2-sphere $F_1 \cup F_3$ separates $C \backslash U$ from B and meets C in a disk. Hence B is cellular.

COROLLARY 5.1. *Let C, J, D, A, and B be as in Theorem 5.1. Suppose that* $\text{Bd}\, C$ *is locally polyhedral* $\text{mod}\, J$ *and that both A and B are cells. Then both A and B are cellular.*

LEMMA 5.1. *Let C be a cellular 3-cell, and let D be a disk on* $\text{Bd}\, C$. *Suppose that* $\text{Bd}\, C$ *is locally polyhedral* $\text{mod}\, D$. *Then D is cellular.*

Proof. Let C_1 be the cube in E^3 which is described by the inequalities $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, and $0 \leqq z \leqq 1$. Let h be a homeomorphism of C onto C_1 which sends D onto the top of the cube (the intersection of C_1 with the plane $z = 1$). For each positive integer i let E_i be the disk which is the intersection of C_1 with the plane $z = 1 - 1/i$. Let $D_i = h^{-1}(E_i)$. Let A_i be the closure of the component of $C \backslash D_i$ which contains D. Then, $\mathrm{Cl}(C \backslash A_i)$ is a 3-cell which is locally polyhedral $\bmod D_i$. Hence A_i is cellular. Since $D = \bigcap_{i=1}^{\infty} A_i$, it follows from Lemma 2.3 that D is cellular.

THEOREM 5.2. *Let C be a 3-cell and let K be a proper subcontinuum of* $\mathrm{Bd}\, C$. *Suppose that K fails to separate the plane and that* $\mathrm{Bd}\, C$ *is locally polyhedral* $\bmod K$. *Then, a necessary and sufficient condition that C be cellular is that K be cellular.*

Proof. The sufficiency follows from Corollary 2.4. If K is cellular, both complementary domains of $\mathrm{Bd}\, C$ are open 3-cells.

To get the necessity, let D be a disk on $\mathrm{Bd}\, C$ such that $K \subset D$ and D is locally polyhedral $\bmod K$. By Lemma 5.1 D is cellular, and by Theorem 4.2 K is cellular.

6. **A cellular arc which contains no tame subarc.** Let α be a wild arc which is the union of two tame arcs β_{11} and β_{12} such that $\beta_{11} \cap \beta_{12} = \{p\}$. This arc is shown by the broken line of Figure I (a). Let $\{\epsilon_i\}_{i=1}^{\infty}$ be a sequence of positive numbers which goes monotonically to zero. Let C_{11} and C_{12} be tame 3-cells such that for $i = 1, 2$,

(1) $\beta_{1i} \subset C_{1i}$,

(2) $\beta_{1i} \cap \mathrm{Bd}\, C_{1i} = \mathrm{Bd}\, \beta_{1i}$,

(3) $C_{11} \cap C_{12} = \{p\}$,

(4) $C_{1i} \subset V(\beta_{1i}, \epsilon_1)$, and

(5) $C_{11} \cup C_{12}$ is wild.

These 3-cells are shown in Figure I (a). By Corollary 2.2, $K_1 = C_{11} \cup C_{12}$ is cellular.

In C_{1i} let α_{1i} be a wild arc joining the endpoints of β_{1i} such that $\mathrm{Int}\, \alpha_{1i} \subset \mathrm{Int}\, C_{1i}$ and such that α_{1i} is the union of two tame arcs. The new arc, $\alpha_{11} \cup \alpha_{12}$, is the union of four tame arcs β_{21}, β_{22}, β_{23}, and β_{24}. These arcs are shown in Figure I (b). For each i ($i = 1, 2, 3, 4$) let C_{2i} be a tame 3-cell such that

(1) $\beta_{2i} \subset C_{2i}$,

(2) $\beta_{2i} \cap \mathrm{Bd}\, C_{2i} = \mathrm{Bd}\, \beta_{2i}$,

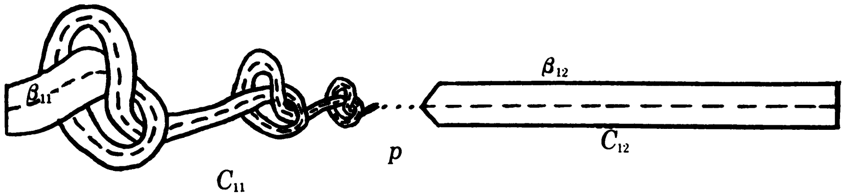
(3) $C_{2i} \cap C_{2j} = \beta_{2i} \cap \beta_{2j}$ for $i \neq j$,

(4) $C_{2i} \subset K_1 \cap V(\beta_{2i}, \epsilon_2)$, and

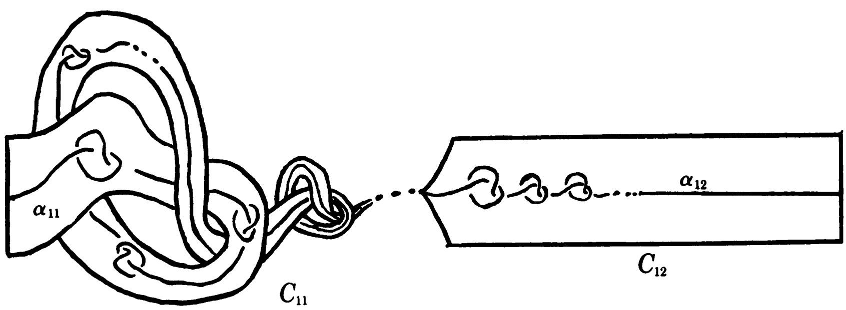
(5) the embeddings of $\bigcup_{i=1}^{4} C_{2i}$ and $\bigcup_{i=1}^{4} \beta_{2i}$ are "equivalent."

Repeated applications of Theorem 2.1 will show that $K_2 = \bigcup_{i=1}^{4} C_{2i}$ is cellular.

In general suppose that $K_i = \bigcup_{j=1}^{2^i} C_{ij}$. Let α_{ij} be an arc joining the endpoints of β_{ij} such that $\mathrm{Int}\, \alpha_{ij} \subset \mathrm{Int}\, C_{ij}$. Suppose that α_{ij} is a wild arc which is the union of two tame arcs. The arc $\bigcup_{j=1}^{2^i} \alpha_{ij}$ is the union



(a)



(b)

FIGURE I

of tame arcs $\beta_{i+1,j}$ $(j = 1, 2, \cdots, 2^{i+1})$. For each j let $C_{i+1,j}$ be a tame 3-cell such that

(1) $\beta_{i+1,j} \subset C_{i+1,j}$,

(2) $\beta_{i+1,j} \cap \mathrm{Bd}\, C_{i+1,j} = \mathrm{Bd}\, \beta_{i+1,j}$,

(3) $C_{i+1,j} \cap C_{i+1,k} = \beta_{i+1,j} \cap \beta_{i+1,k}$ for $j \neq k$,

(4) $C_{i+1,j} \subset K_i \cap V(\beta_{i+1,j}, \epsilon_{i+1})$, and

(5) the embeddings of $\bigcup_{j=1}^{2^{i+1}} C_{i+1,j}$ and $\bigcup_{j=1}^{2^{i+1}} \beta_{i+1,j}$ are "equivalent."

Repeated applications of Theorem 2.1 will show that $K_{i+1} = \bigcup_{j=1}^{2^{i+1}} C_{i+1,j}$ is cellular.

This process can be done so that for some number η, $0 < \eta < 1$, the diameter of $C_{i+1,j} < \eta(\max_{1 \leqq k \leqq 2^i}$ diameter of $C_{ik})$ for all i and $1 \leqq j \leqq 2^{i+1}$. Then $\beta = \bigcap_{i=1}^{\infty} K_i$ is an arc, and by Lemma 2.3, β is cellular.

For each i and j, the subarc $\beta \cap C_{ij}$ is equivalent (with respect to embeddings) to β and hence wild. Let γ be a subarc of β. For i sufficiently large, there is a j such that $\beta \cap C_{ij} \subset \gamma$. Hence γ is wild.

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